Impossibility of Probabilistic splitting of quantum information

I.Chakrabarty [1 2],[*] B.S.Choudhury [2]

[1] Heritage Institute of Technology, Kolkata, India

[2] Bengal Engineering and Science University, Howrah, India

Abstract

We know that we cannot split the information encoded in two non-orthogonal qubits into complementary parts deterministically. Here we show that each of the copies of the state randomly selected from a set of non orthogonal linearly independent states, splitting of quantum information can not be done even probabilistically. Here in this work we also show that under certain restricted conditions, we can probabilistically split the quantum information encoded in a qubit.

1 Introduction :

In quantum information theory understanding the limits of fidelity of different operations has become an important area of research. Noticing these kind of operations which are feasible in classical world but have a much restricted domain in quantum information theory started with the famous 'no-cloning' theorem [1]. The theorem states that one cannot make a perfect replica of a single quantum state. Later it was proved that one cannot clone two non-orthogonal quantum states [2]. However this does not rule out the possibility of producing approximate cloning machines [3-5]. Even though deterministic

*E-mail: indranilc@indiainfo.com

cloning is not possible, a probabilistic cloning machine can be designed , which will clone the input states with certain probabilities of success [6]. Though the unitarity of quantum mechanics prohibits accurate cloning of non orthogonal quantum states , but such a class of states secretly chosen from a set containing them can be faithfully cloned with certain probabilities if and only if they are linearly independent. Basically Quantum copying machine can be divided into two classes (i)deterministic quantum copying machine (ii) probabilistic quantum copying machine. The first type of quantum copying machine can be divided into two further subclasses: (i) State dependent quantum cloning machine , for example, Wooters-Zurek (W-Z) quantum cloning machine [1], (ii) Universal quantum copying machine, for example, Buzek-Hillery (B-H) quantum cloning machine [2].

Pati and Braunstein introduced a new concept of deletion of an arbitrary quantum state and shown that an arbitrary quantum states cannot be deleted. This is due to the linearity property of quantum mechanics. Quantum deletion [7,8] is more like reversible 'uncopying' of an unknown quantum state. The corresponding no-deleting principle does not prohibit us from constructing the approximate deleting machine [16]. J. Feng et.al. [9] showed that each of two copies of non-orthogonal and linearly independent quantum states can be probabilistically deleted by a general unitary-reduction operation. Like universal quantum cloning machine, D'Qiu [15] also constructed a universal deletion machine but unfortunately the machine was found to be non-optimal in the sense of fidelity. A universal deterministic quantum deletion machine is designed in an unconventional way that improves the fidelity of deletion from 0.5 and takes it to 0.75 in the limiting sense [19]. Many other impossible operations generally referred as 'General Impossible operations' [10] can not be achieved successfully with certainty , but one can carry out these operations at least probabilistically with certain probability of success [11].

Recently 'no-splitting' theorem [12] is an addition to these set of no-go theorems. The theorem states that the quantum information of a qubit cannot be split into complementary parts. The no-splitting theorem, can be mathematically stated as whether the two real parameters (θ, ϕ) can be split into two complementary qubits as follows: $L(|A(\theta,\phi)\rangle|B\rangle = |A(\theta)\rangle|B(\phi)\rangle)$? The answer is no. The linearity of quantum mechanics [12] as well as the unitarity of quantum mechanics doesn't allow the splitting of the

information contained inside a qubit.

Similarly 'partial erasure of quantum information' [13] is another operation which is not possible in quantum world. The 'no-splitting' theorem can also be obtained as a special case of 'no-partial erasure' of quantum information theorem. It remains interesting to see that whether we can split quantum information at least probabilistically. In this work we try to find out whether we can split the information in a qubit with a certain probability of success. We show that we cannot even probabilistically split the quantum information inside a qubit. Here we will find that unlike cloning and deletion if non orthogonal quantum states are secretly chosen from a set then there exists no such transformation that will split the quantum information of a qubit with certain probability of success. However we also show that under certain restricted conditions the splitting of quantum information will be possible.

2 Probabilistic Quantum information splitting:

The quantum 'no-splitting' theorem [12] says that exact splitting of quantum information encoded in two non orthogonal states cannot be done. Nevertheless, it does not get rid of the possibility of splitting the quantum states with certain probabilities or in other words one may ask that is there any unitary reduction process which will split the information encoded in two non-orthogonal states $|\psi_i(\theta_i,\phi_i)\rangle$ and $|\psi_j(\theta_j,\phi_j)\rangle$ secretly chosen from a set $S = \{|\psi_1(\theta_1,\phi_1)\rangle, |\psi_2(\theta_2,\phi_2)\rangle, ...|\psi_n(\theta_n,\phi_n)\rangle\}$. Here $|\psi_i(\theta_i,\phi_i)\rangle = \cos(\frac{\theta_i}{2})|0\rangle + \sin(\frac{\theta_i}{2})e^{\imath\phi_i}|1\rangle$ are the quantum states represented as a point on the Bloch sphere (where $\imath = \sqrt{-1}$). Let us consider two systems A and B . Now each of the states of the set S can be taken as a input state of the system A. Let us consider a unitary evolution U and measurement M, which together yield the following evolution

$$|\psi_i(\theta_i,\phi_i)\rangle|\Sigma\rangle \longrightarrow^{[U+M]}\longrightarrow |\psi_i(\theta_i)\rangle|\Sigma(\phi_i)\rangle \tag{1}$$

where $|\Sigma\rangle$ is the input state of the ancillary system B. Both the systems A and B are described by a N dimensional Hilbert space with $N \geq n$.

To continue with the argument of the above statement, a probe P with n_p ($n_p \geq n+1$) dimensional Hilbert space is introduced , where $\{|P_0\rangle, |P_0\rangle, ..., |P_n\rangle\}$ are n+1 orthonormal states of the probe. Now let us introduce a unitary operator U whose action on the tensor products of the Hilbert spaces associated with the system A,B and probe P is given by

$$\begin{aligned} U(|\psi_i(\theta_i,\phi_i)\rangle|\Sigma\rangle|P_0\rangle) &= \sqrt{\gamma_i}|\psi_i(\theta_i)\rangle|\Sigma(\phi_i)\rangle|P_0\rangle \\ &+ \sum_{j=1}^{n} c_{ij}|\Phi_{AB}^{(j)}\rangle|P_j\rangle, \quad (i=1,2,..,n) \end{aligned} \tag{2}$$

where $|\psi_i(\theta_i)\rangle = \cos(\frac{\theta_i}{2})|0\rangle + \sin(\frac{\theta_i}{2})|1\rangle$ and $|\Sigma(\phi_i)\rangle = \frac{1}{\sqrt{2}}[|0\rangle + e^{\imath\phi_i}|1\rangle]$. Here, $\{|\Phi_{AB}^{(j)}\rangle\}$ (j=1,...,n) are normalized states of the composite system AB, and these states are not necessarily orthogonal. After the unitary evolution , the measurement is made on the probe P . The attempt made for splitting the information into constituent parts will succeed with γ_i probability of success if the measurement outcome of the probe is P_0.
We start here by showing that unlike probabilistic cloning and deletion , probabilistic splitting of quantum information will not be possible for both linearly dependent and independent states secretly chosen from the set S. Let us introduce a theorem.

Theorem1 : **The states which are secretly chosen from the set** $S = \{|\psi_1(\theta_1,\phi_1)\rangle, |\psi_2(\theta_2,\phi_2)\rangle, ...|\psi_n(\theta_n,\phi_n)\rangle\}$ **can be probabilistically split (realization of the unitary evolution) if the states are linearly independent** .
Proof : Consider an arbitrary state which can be expressed as the linear combination of the states in the set S.

$$|\psi(\theta,\phi)\rangle = \sum_{i=1}^{n} d_i|\psi_i(\theta_i,\phi_i)\rangle \tag{3}$$

The unitary transformations of the arbitrary linearly dependent state vector is given by,

$$U(|\psi(\theta,\phi)\rangle|\Sigma\rangle|P_0\rangle) = \sqrt{p}|\psi(\theta)\rangle|\Sigma(\phi)\rangle|P_0\rangle + c|\Phi_{AB}\rangle|P_{\perp 0}\rangle \tag{4}$$

But, if we consider the action of the unitary transformation defined in (2) on the linear combination of the state vectors belonging to the set S, then the resultant is given by,

$$U(\sum_{i=1}^{n}|\psi_i(\theta_i,\phi_i)\rangle|\Sigma\rangle|P_0\rangle) = \sum_{i=1}^{n}\sqrt{p_i}d_i|\psi_i(\theta_i)\rangle|\Sigma(\phi_i)\rangle|P_0\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n} d_i c_{ij}|\Phi_{AB}^{(j)}\rangle|P_j\rangle \tag{5}$$

Now it is clearly evident that the final states (4) and (5) are different quantum states . Since the state $|\psi(\theta,\phi)\rangle$ is a linear combination of the state vectors $|\psi_i(\theta_i,\phi_i)\rangle$ belonging to the set S, the linearity of quantum mechanics is prohibiting the existence of probabilistic quantum information splitting machine. Therefore the unitary evolution given by (2) exists for any set secretly chosen from the set S only if the states belonging to the set S are linearly independent.

The interesting part is that the converse of the theorem is not true. The converse statement of the *Theorem1* is given as follows :
If the quantum states $|\psi_i(\theta_i,\phi_i)\rangle \quad (i=1,..n)$ in the set S are linearly independent then the unitary evolution (2) exists.

However we will find that this will not hold in general. Interestingly we will also see that there are few particular cases for which the converse is true, and consequently the information splitting is possible.
In other words we can say that if the states chosen secretly are linearly independent, the unitary evolution (2) will not hold with positive definite matrices $\sqrt{\Gamma}$, consequently the physical process described by (1) is not going to be realized. In order to verify the existence of the unitary evolution (2) we must introduce the following lemma.

Lemma1: If two sets of states $\{|X_1\rangle,|X_2\rangle,.....,|X_n\rangle\}$ and $\{|\tilde{X}_1\rangle,|\tilde{X}_2\rangle,.....,|\tilde{X}_n\rangle\}$ satisfy the condition

$$\langle X_i|X_j\rangle = \langle\tilde{X}_i|\tilde{X}_j\rangle \quad (i=1,...n; j=1,...n) \tag{6}$$

there exists a unitary operator U to make $U|X_i\rangle = |\tilde{X}_i\rangle \quad (i=1,...,n)$

The $n\times n$ inter-inner products of equation (2) yield the matrix equation

$$D = \sqrt{\Gamma}GH\sqrt{\Gamma}^{+} + CC^{+} \tag{7}$$

where $D=[\langle\psi_i(\theta_i,\phi_i)|\psi_j(\theta_j,\phi_j)\rangle]$, $G=[\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle]$, $H=[\langle\Sigma(\phi_i)|\Sigma(\phi_j)\rangle]$ and $C=[c_{ij}]$. The diagonal efficiency matrix Γ is defined by $\Gamma = diag(\gamma_1,\gamma_2,...,\gamma_n)$, hence

$\sqrt{\Gamma} = \sqrt{\Gamma}^+ = diag(\sqrt{\gamma_1}, \sqrt{\gamma_2}, ..., \sqrt{\gamma_n})$. Now if *lemma1* clearly shows that the equation (7) is satisfied with a diagonal positive-definite matrix Γ , then the unitary evolution (2)will hold, consequently the physical process (1) can be realized in physics.
To show that there is a diagonal positive definite matrix Γ to satisfy equation (7), first we need to show that the matrix D is positive-definite.
We introduce a Lemma to show that D is positive-definite.

Lemma2: If n states $[|\psi_1(\theta_1, \phi_1)\rangle, |\psi_2(\theta_2, \phi_2)\rangle, ...|\psi_n(\theta_n, \phi_n)\rangle]$ are linearly independent, then the matrix $D = [\langle\psi_i(\theta_i, \phi_i)|\psi_j(\theta_j, \phi_j)\rangle]$ is positive definite.

Proof: For any arbitrary n-vector $B = (b_1, b_2, ...b_n)^T$, the quadratic form B^+DB can be expressed as

$$B^+DB = \langle\Psi|\Psi\rangle = \||\Psi\rangle\|^2 \tag{8}$$

where

$$|\Psi\rangle = b_1|\psi_1(\theta_1, \phi_1)\rangle + b_2|\psi_2(\theta_2, \phi_2)\rangle + ... + b_n|\psi_n(\theta_n, \phi_n)\rangle \tag{9}$$

Since we know that states $[|\psi_1(\theta_1, \phi_1)\rangle, |\psi_2(\theta_2, \phi_2)\rangle, ...|\psi_n(\theta_n, \phi_n)\rangle]$ are linearly independent, the state $|\Psi\rangle$ does not reduce to zero for any n-vector B and its norm will always remain positive. Hence from definition D is positive-definite.

But the matrix $L = D - GH$ is not a Hermitian matrix in general.This is because the matrix G is a real symmetric matrix while the matrix H is a Hermitian matrix, and we know that the product of the real symmetric matrix and the hermitian matrix is not going to give a resultant hermitian matrix all the times. If we observe the matrix G we see that the (i,j) th element of the matrix is given by the inner product $\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle = \cos(\frac{\theta_i}{2})\cos(\frac{\theta_j}{2}) + \sin(\frac{\theta_i}{2})\sin(\frac{\theta_j}{2})$, which is a real quantity. Here we clearly see that in the matrix G (i,j)th element is equal to (j,i) th element , with one as the principal diagonal entries .Hence the matrix G is a real symmetric matrix
In the matrix H the (i,j) th entry is given by, $\langle\Sigma(\phi_i)|\Sigma(\phi_j)\rangle = \frac{1}{2}[1 + e^{\imath(\phi_j - \phi_i)}]$ which is a complex quantity and here we see that the (j,i) th entry of the matrix H is conjugate of

the (i,j) th entry, with one as principal diagonal entries. From here we conclude that the matrix H is a Hermitian matrix.

In general, the principal diagonal elements of the matrix $L = D - GH$ is given by,
$L_{ii} = \langle\psi_i(\theta_i,\phi_i)|\psi_i(\theta_i,\phi_i)\rangle - \gamma_i(\langle\psi_i(\theta_i)|\psi_i(\theta_i)\rangle + \sum_{j=1,j\neq i}^{n}\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle\langle\Sigma_j(\phi_j)|\Sigma_i(\phi_i)\rangle)$.
The off diagonal elements of the matrix L is given by,
$L_{ij} = \langle\psi_i(\theta_i,\phi_i)|\psi_j(\theta_j,\phi_j)\rangle - \sqrt{\gamma_i\gamma_j}(\sum_{k=1}^{n}\langle\psi_i(\theta_i)|\psi_k(\theta_k)\rangle\langle\Sigma_k(\phi_k)|\Sigma_j(\phi_j)\rangle)$.
Unless the matrix L is hermitian , we cannot have the corresponding quadratic form. For the matrix L to be Hermitian , we must have
(i) $L_{ij} = L_{ji}^*$, (where L_{ji}^* is the conjugate of L_{ji})
(ii) L_{ii} will be a real quantity.
In general the matrix L is not Hermitian as the elements L_{ii} are in general complex quantities, as a consequence of which we don't have the corresponding quadratic form, and henceforth there arise no question for showing L as a positive definite matrix. However under certain conditions we can show the matrix L to be positive definite.
To show the matrix L to be hermitian we must have to show only the principal diagonal elements L_{ii} are real, as $L_{ij} = L_{ji}^*$. Now the diagonal elements L_{ii} are real iff $\sum_{j=1,j\neq i}^{n}\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle Im[\langle\Sigma_j(\phi_j))|\Sigma_i(\phi_i)\rangle]) = 0$ and hence the principal diagonal elements reduces to $L_{ii} = \langle\psi_i(\theta_i,\phi_i)|\psi_j(\theta_j,\phi_j)\rangle - \gamma_i(\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle + \sum_{j=1,j\neq i}^{n}\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle$ $Re[\langle\Sigma_j(\phi_j)|\Sigma_i(\phi_i)\rangle])$. Now if we consider the corresponding quadratic form of the matrix L,

$$XLX^t = (x_1, x_2, ..., x_n)[L_{ij}](x_1, x_2, ..., x_n)^t = \sum_{i=1}^{n} L_{ii}x_i^2 + \sum_i\sum_j L_{ij}x_ix_j \tag{10}$$

Now the matrix L is positive definite only when the above expression (10) is positive. This is possible only under the following conditions:
(i)$L_{ii} = \langle\psi_i(\theta_i,\phi_i)|\psi_i(\theta_i,\phi_i)\rangle - \gamma_i(\langle\psi_i(\theta_i)|\psi_i(\theta_i)\rangle + \sum_{j=1,j\neq i}^{n}\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle\langle\Sigma_j(\phi_j)|\Sigma_i(\phi_i)\rangle)) > 0$
(ii) $\sum_{i=1}^{n} L_{ii}x_i^2 > \sum_i\sum_j L_{ij}x_ix_j \Longrightarrow \sum_i[1-\gamma_i-\gamma_i\sum_{j=1,j\neq i}^{n}\langle\psi_i(\theta_i)|\psi_j(\theta_j)\rangle\langle\Sigma_j(\phi_j)|\Sigma_i(\phi_i)\rangle]x_i^2 > \sum_i\sum_j[\langle\psi_i(\theta_i,\phi_i)|\psi_j(\theta_j,\phi_j)\rangle - \sqrt{\gamma_i\gamma_j}(\sum_{k=1}^{n}\langle\psi_i(\theta_i)|\psi_k(\theta_k)\rangle\langle\Sigma_k(\phi_k)|\Sigma_j(\phi_j)\rangle)]x_ix_j \quad \forall x_i, x_j$. If the above conditions are satisfied, then the matrix L will be a positive definite matrix.

As a consequence of which we can say that the equation (7) will be satisfied by positive definite matrix Γ. Hence under these conditions the converse of the *Theorem1* will be satisfied, henceforth unitary evolution (2) will hold and equation (1) can be realized in physics. This clearly indicates that there are certain class of states on Bloch sphere satisfying the above conditions for which the information splitting will be possible.

3 Conclusion

In summary we can say that there is no possibility of splitting the quantum information either deterministically or probabilistically. The result obtained here is interesting in the sense that it will help us to understand and to classify the impossible operations in quantum information theory more specifically. As a consequence of which one can make a comment that splitting of quantum information are different from cloning and deletion in the sense that these operations unlike cloning and deletion cannot be achieved even probabilistically. However this doesn't rule out the probabilistic quantum information splitting of certain class of states under certain restricted conditions. This also doesn't rule out the possibility of approximate splitting of quantum information .

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